Studies on Grübler's formula for mobility and connectivity

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Abstract—Kinematic synthesis is a challenging task for a designer to devise a mechanism that can satisfy desired kinetic as well as kinematic characteristic of the overall linkage. Both analysis and synthesis of kinematic chains are very important from the view point of mechanical engineering design and this have attracted the sustained efforts of many researchers to study various aspects of mechanisms. Determination of mobility or degree of freedom is the most critical phase for designing a mechanism. The very first work in the field of mobility was done in the mid 19th century by P L Chebychev, followed by Sylvester and Grübler. From then, lot of contributions has been made, but so far, no single method suits to all the classical and the modern mechanisms. Continuous efforts have been made to find general and unique formula for a quick calculation of mobility that may be applicable to any rigid body mechanism. Among these research work, Grübler-Kutzbach criterion has been proved to be the most versatile which can reliably work with most of the planar and spatial mechanism. In this paper, the Grübler's formula for mobility and connectivity has been reviewed, and its application for motion analysis of robotic hand along with its concerns and limitations of using this method, has been studied.

Index Terms—Degree of freedom, Mobility, Connectivity, Grübler's formula, Robotic hand.

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1 INTRODUCTION

EVERY machine is a combination of one or several mechanisms. In order to get the desired output(actuation or control) of a machine or desired motion of a mechanism, one goes for mechanism design and synthesis. Kinematic synthesis can be defined as the systematic design of mechanisms for a given performance (Erdman & Sandor, 1991). It is a systematic approach which covers the selection of suitable mechanism type according to the given path, calculation of the necessary dimensions, force and velocity of the mechanism and its elements, active workspace and form generation capacity. The very first step to mechanism design is the determination of mobility, which gives the motion capability of the system. According to IFToMM terminology, the mobility or the degree of freedom is defined as the number of independent co-ordinates needed to define the configuration of a kinematic chain or mechanism[1].

The work on mobility enunciated in the mid-19th century by Chebychev, followed by Sylvester, Grübler, Somov and Hochman and many others till 20th century to generalize methods for the determination of the mobility of any rigid body mechanism. Various formulas and approaches were derived and presented in the literature by Koenigs, Grübler, Malytsheff, Kutzbach, Artobolevski and many others[2].

Chebychev was the first to present the mechanism mobility in mathematical form. He developed the first formula for the calculation of the number of independent variables in a mechanism, in the late nineteeth century.

Chebychev expressed his formula in the form:

$$3n-2(p_0+p_n)=1 \qquad (1)$$

where $3n$ is the number of variables required to describe the position and the orientation of the n kinematic bars in the plane. $2(p_0 + p_n)$ is the number of constraint equations imposed by the $p(p_0 + p_n)$ revolute joints of the mechanism that can be adjacent (p_0) or non-adjacent (p_n) to the fixed base. It is known that each revolute joint introduces two constraint equations in a planar mechanism. Chebychev applied this formula to elementary planar mechanisms with $p_0 = 2$ and to complex planar mechanisms ($p_0 = 3$) having only revolute joints and one degree of mobility. Hence, for a planar mechanism with one degree of freedom at the joints (e.g. helical, prismatic and revolute joints), eq (1), can be expressed for mobility as

$$M=3n-2(p_0+p_n) \qquad (2)$$

In 1874, Sylvester presented a modified form of Chebychev as a structural condition for one degree of freedom pin-connected planar mechanisms:

$$3m-2p-4=0 \qquad (3)$$

where, m is the total number of elements of a mechanism (both fixed and kinematic elements).

After Sylvester, Grübler, in 1883 presented a structural condition for one degree of freedom planar mechanisms[3] identical with the Eq. (3). Later, he extended this structural condition to one degree of freedom spatial complex mechanisms with helical joints[4]:

$$5h-6m+7=0 \qquad (4)$$

where h is the total number of helical joints.

A wide scope for research was opened afterward, in finding the most suitable and globally applicable formula for mobility of all simple and complex, planar and spatial, open and closed loop mechanisms. Nearly, all methods, discovered till today, have some flaws and fail to be applicable for all mechanisms.

2 MARTIN FÜRCHTEGOTT GRÜBLER

Martin Grübler was a German Mechanical engineer. He was born in Meerane in 1851 and studied in Dresden and Leipzig (1870-1880).1880 Technical Head teacher of examination in

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Dresden. He qualified as a lecturer in mechanics and taught at the ETH Zurich. In 1886, he became a professor of mechanics at the Polytechnic Riga (Imperial Russia) and professor of Applied Mechanics at TH Dresden in 1900. He remained there as a director of the collection of Applied Mechanics and Graphical Statics from 1910 to 1920. In 1917, Grübler published his "Getriebelehre - a theory of forced run and the planar mechanisms" (the Chebychev-Grübler-Kutzbach criterion is named for him here). In 1927, he received an honorary doctorate from the University of Riga and an honorary doctorate award of the University of Giessen (German National Library) [German National Library]

Today, in general, mobility criterion is based on Grübler formula (1917), and its extension modifications, known by Kutzbach and modified Kutzbach formula.

3 BASIC TERMINOLOGIES

3.1 Mobility and connectivity

A mechanism's mobility is the total degrees of freedom which need to be controlled in the mechanism for every link to be in a specific position. According to IFToMM terminology, the mobility is defined as the number of independent co-ordinates needed to define the configuration of a kinematic chain or mechanism[1]. This is similar to degrees of freedom of mechanisms and is used interchangeably. It is also interpreted as the number of independent actuator required to move the mechanism in the desired manner.

The overall motion capability of the chain (selection of fixed link and power link or actuator) depends on the category which the multi degree of freedom chain falls in. The different types of mobility for multi degree of freedom (f) chain include: non-degenerate mobility (all sub-chains or loops with degree of freedom f > 0), fractionated mobility (a link called separation link divides the whole chain into two sub chains with f1 and f2 degree of freedom such that f = f1 + f2), partial mobility (at least one sub-chain has f' degree of freedom such that 0 ≤ f' < f), and total mobility (all sub-chains have degrees of freedom f'≥ f) [5][6][10].

In mechanism theory, the connectivity Cij between two links i and j of a kinematic chain is the relative mobility between links i and j[11]. It can be easily determined, whether a mechanism can perform the desired task (i.e. the output link has motion parameters as per requirement) relative to the fixed link, or not from its connectivity and not mobility.

A conceptual definition for connectivity had been introduced by Phillips as a concept of joint in the bag equivalence, wherein a flexible black bag acts as an equivalent unknown for the joint between links i and j, hidden in the bag[12]. Many tools are available for determining connectivity in a mechanism. Contributions to this field are made by Hunt 1978, Tischler et al. 2001, Tischler et al. 1995, Liberati and Belfiore 2006, Belfiore and Benedetto 2000, Roth and Shoham and many others, which drives the efforts to find an algorithm for the numerical calculation of connectivity.

Tischler et al. 1995 introduced new concepts: the variety of a kinematic chain and the minimal sets of kinematic chains. B Roth and M shoham had introduced the concept of link connectivity and adopted a novel method of producing modified graph to facilitate computation of link connectivity. Connectivity matrix is calculated from connectivity matrix obtained by the correspondence between kinematic chains and graphs.

3.2 Grüblers formula

Although, dozens of work have been done for determination of mobility since the mid nineteenth century, but these traditional methods do not suit most of the modern as well as some classical mechanisms, especially in the field of robotics. Up to now, after looking at the drawbacks of these formulae, one can eventually come up with a conclusion that the most appropriate formula which can work with most plane and spatial mechanisms, is Kutzbach-Grübler formula[7]. The formula was first derived by Grübler and later modified by Kutzbach. Grübler formulated a relation of mobility, in terms of number of links (moving and fixed) and number of joints. Later, Kutzbach found it that any mechanism can have just one ground link and hence, he came up with a new formula similar to that of Grübler with the only replacement of the variable representing the number of ground links by 1.

Grübler and Kutzbach formula allows the basic calculation for a mechanism's mobility, which we often call its degrees of freedom. For an open kinematic chain with n links joined by j number of joints, the formula for mechanism's mobility, M is:

$$M \geq \lambda(n-1) + \sum_{i=1}^{j} u_i \qquad (5)$$

λ = degrees-of-freedom in space in which mechanism functions (λ is 6 for spatial mechanism and 3 for planar mechanism)

u_i is the number of constraints imposed by joint i which is related to the freedom of the i^{th} joint, f_i as;

$$u_i + f_i = \lambda \qquad (6)$$

Eq. (5), therefore becomes

$$M \geq \lambda(n-j-1) + \sum_{i=1}^{j} f_i \qquad (7)$$

In general, we use the equality sign to determine M, the greater-than is used only when the mechanism has special proportions[8].

For a closed kinematic chain with l loops (where each loop increases the excess of joints over link by 1), the mobility equation reduces to:

$$M = \sum_{i=1}^{j} f_i - \lambda l \qquad (8)$$

For eg. if we apply this formula to Watt's chain, we get M=1 by Planar Grübler's formula and M = -2 by spatial Grübler's formula.

Grübler's Criterion is valid in nearly all planar and spatial mechanisms as long as there are no redundant joints. A redundant joint is one that is unnecessary because other joints can provide the needed position and/or orientation. Redundant joints can generate passive degrees-of-freedom, which must be subtracted from Grübler's equation to get;

$$M \ge \lambda\left(n - j - 1\right) + \sum_{i=1}^{j} f_i - f_p \tag{9}$$

Passive freedoms are introduced intentionally to supply actuations, facilitate easy assembly and compensate for errors due to manufacturing inaccuracies.

4 APPLICATION OF MOBILITY CONCEPT TO ROBOHAND

Mechanical design and manipulation of robotic hands has been an active area of research for the past few decades, due to rapid demands of robots in variety of applications spreading its wings in industrial, medical, military, household and many other uncountable applications. Industrial applications of robots include material handling, transportation between stations (in the form of AGVs), assembly, packaging, testing, etc.

Attempts are made to design a simple and compact but highly flexible and dynamic robohand with excellent performance. Upper limb prosthetics as an application of dexterous artificial hand design and manipulation has gained wide attention of numerous researchers. The most common issue is hand kinematics (dealing with motion characteristics and hand control including manipulation and grasping: grasp pre-shaping and grasp synthesis operations). The manipulation and grasping capability of multi-fingered robotic hands has been addressed through the concepts of mobility and connectivity of multi-fingered hands, see fig. 1 (Mason and Salisbury,1985).

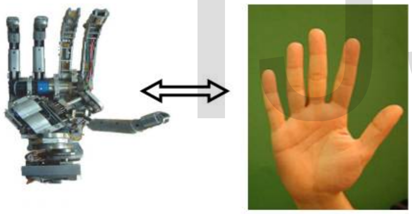

Fig. 1 MANUS hand as an imitation of Human hand

The primitive goal for every research is to come up with a robohand that resembles the human hand as close as possible. Table 1. shows the comparison of SRMSCET hand with the human hand. The imitation of human hand will make the robohand capable of carrying out advanced function, useful as the end effecters, in the fields, such as tele-operation with master-slave system. However, the task of reproducing the motion of the human hand is not that simple, due to current technical constraints on actuators, sensors and control means.

TABLE 1. COMPARISON OF MOBILITY OF HUMAN HAND WITH SRMSCET HAND

Category	DOF	Wrist Mobility
Human hand	22	3
SRMSCET HAND	15	2 (if the wrist is considered to be a spherical joint, at the base plate)

In context of hand taxonomy, the manipulative movements of the hand can be classified into extrinsic and intrinsic movements. Extrinsic movements are the motion of thegrasped object by displacement of the hand as a whole viz. related to dynamic manipulation of the object, while intrinsic movements define the motion of the object within the hand, categorized under object grasping. In fig. 2, the six basic grasping modes of human hand are shown. Setup in 1998, the MANUS-HAND projectwith an overall objective of developing a multifunctional robotic hand prosthesis with enhanced mobility aims at achieving these six basic modes of grasp [13].

In designing multi-fingered dexterous robohands with dynamic manipulability, one has to start with degrees of freedom determination and mobility and connectivity analysis. In this regard, Kutzbach-Grübler formula has been used extensively to determine possible structures for manipulators, hands, and end effectors. To enhance the manipulability one needs to increase degree of freedom of overall finger, hand and arm mechanism. However, we need to be cautious, since in devices like manipulators we are not generally concerned with the mechanism's mobility, but rather the number of degrees-of-freedom between two specific links usually, the freedom between the ground link and the end-effector link, or in other words, connectivity. If two links are jointed together the connectivity equals the freedom for that joint. For non-adjacent links the connectivity is upper bounded by the mechanism's mobility.

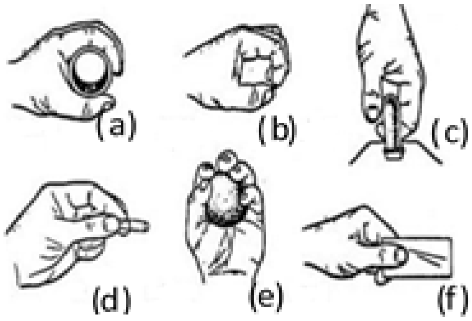

Fig. 2. (a) Cylindrical or power grasp (b) Precision grasp (c) hook prehension (d) tip grasp (e) spherical grasp (f) lateral grip [13]

The following examples shows the application of Grübler formula for calculating degrees of freedom in robohand.

Figure 3 - ABB 6-axis robot

For ABB IRB 4400 robotic arm (fig. 3), Grübler equation (eq. 7) gives,

$M = 6\,(7 - 1) - 6\,(5) = 6$

Similarly, we can use mobility formula of closed loop chains, eq.(4) to calculate degree of freedom for parallel robots.

For eg. Stewart-Gough platform shown in fig. 4, has the following geometrical parameters;

$$\lambda = 6;\ n = 14;\ j = 18;\ \text{and}\ f_p = 6$$

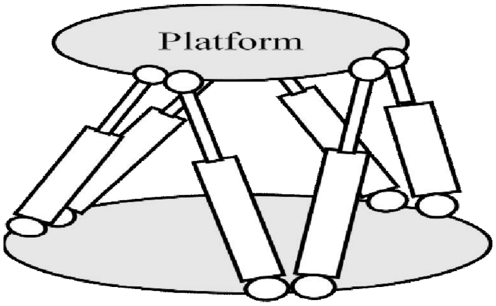

Fig. 4 Stewart-Gough platform

This gives,

$$M = 6(14 - 18 - 1) + (12x3 + 6) - 6 = 6,$$

which is correct.

In parallel manipulators, J > DOF → J −DOF joints are passive. Example: In a simple 4-bar mechanism, J = 4 and DOF = 1. This implies that only one joint is actuated and three are passive. Similarly, in 3-RPS manipulator, J = 9 and DOF = 3 → 6 joints are passive. Passive joints can be multi- degree-of-freedom joints. In 3-RPS manipulator, three- degree-of-freedom spherical (S) joints are passive. In a Stewart platform, the S and U joints are passive.

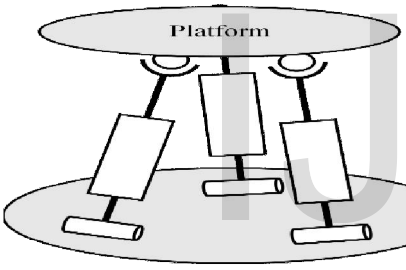

Fig. 5. 3 RPS parallel manipulator

Moreover, the overall kinematics of the multi-fingered hand can be defined according to the required task, i.e., when dynamic manipulation is required, a connectivity of C = 6 gives the maximum dexterity; when grasping is required, a connectivity C ≤ 0 ensures that no degree of freedom is left to the grasped object.

Among the so far developed robohands, in the category of DRHs and APHs, the DRH utilizes the mechatronic technique of reproducing dexterous human hand's manipulation. On the basis of drive position (inside or outside the hand), the DRH can be mainly divided into two categories: intrinsic actuation pattern (IAP) or extrinsic actuation pattern (EAP). Some representative DRHs with certain specifications are shown in Table 2.

TABLE 2. MOBILITY OF SEVERAL POPULAR ROBOTIC HANDS [9]

S.No.	Name	Fingers	DOF	Actuation configuration
1	Okada hand	3	11	Extrinsic
2	DLR-I hand	4	12	Intrinsic
3	DLR-II hand	4	13	Intrinsic
4	UB-II hand	3	11	Extrinsic
5	UB-III hand	5	16	Extrinsic
6	Robonaut hand	5	14	Extrinsic
7	ZJUT hand	5	20	Extrinsic
8	DLR/HIT I	4	13	Intrinsic
9	DLR/HIT II	5	15	Intrinsic

4 CONCLUSION

Applicability of mobility in kinematic design of linkage makes it a critical area of study, starting from the late nineteenth century till now. Mobility and connectivity of linkages is not a new area of research, but still has not fully grown up. The problem of identifying the correct mobility of mechanisms has been a source of concern for many researchers. Sustained efforts have been made to find general formula for a quick calculation of mobility of any rigid body mechanism but a unique and dynamic solution to this problem yet to be found out that can be fit to any classical mechanisms, especially recent parallel dexterous robotic arms. Many of these methods are reducible to the same originated formula. Among the so far developed equations for mobility, Kutzbach-Grübler criterion, with the correction for passive degree of freedom, is the most suitable one for both planar and spatial linkage. Like many others, Grübler equation only relates the number of links and joints to mobility without taking into consideration the links dimension and other geometric features. This may result into certain exceptions of incorrect result. Hence, a better methodology has to be framed in structural analysis for determination of motion characteristic of any mechanisms.

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